Temperature dependence of the superheating field for superconductors in the high-κ London limit

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We study the metastability of the superheated Meissner state in type II superconductors with $\kappa \gg 1$ beyond Ginzburg-Landau theory, which is applicable only in the vicinity of the critical temperature. Within Eilenberger's semiclassical approximation, we use the local electrodynamic response of the superconductor to derive a generalized thermodynamic potential valid at any temperature. The stability analysis of this functional yields the temperature dependence of the superheating field. Finally, we comment on the implications of our results for superconducting cavities in particle accelerators.

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I. INTRODUCTION

The Meissner effect—the expulsion of a weak magnetic field from a bulk superconductor—is one of the hallmark of superconductivity. As the field is increased, the response of a superconductor depends on the value of the Ginzburg-Landau (GL) parameter $\kappa = \lambda/\xi$, where λ and ξ are the magnetic-field penetration depth and the superconducting coherence length, respectively. Type I superconductors ($\kappa < 1/\sqrt{2}$) usually turn normal at the thermodynamic critical field H_c, but superconductivity can be maintained, as a metastable state, up to the superheating field $H_{\mathrm{sh}} > H_c$. As for type II superconductors ($\kappa > 1/\sqrt{2}$), above the first critical field H_{c1} their stable state is characterized by the presence of vortices, and superconductivity persists up to the second critical field H_{c2}. However, since the work of Bean and Livingston[1] it is known that an energy barrier at the surface impedes the penetration of vortices into the bulk, making it possible for the Meissner state to exist as a metastable state up to a superheating field $H_{\mathrm{sh}} > H_{c1}$. Therefore H_{sh} is a characteristic property of both types I and II superconductors. In this paper we consider the temperature dependence of the superheating field in strong type II superconductors with $\kappa \gg 1$—i.e., in the London (or local) limit.

Over the years the issue of the stability of the superheated Meissner state has received much attention. The simplest system in which this problem can be studied is a clean superconductor occupying a half space with a magnetic field applied parallel to the surface. For this system in the strong type II limit, and assuming that the instability is due to fluctuations in the direction perpendicular to the surface (i.e., one-dimensional fluctuations), de Gennes[2] calculated the superheating field $H_{\mathrm{sh}} = H_c$ near the critical temperature T_c. If the instability signals the penetration of vortices, however, the relevant fluctuations can be expected to vary along two dimensions while preserving translational invariance along the field direction. Galaiko[3] showed that this is indeed the case and near T_c the actual superheating field is smaller than that found by de Gennes, $H_{\mathrm{sh}} \simeq 0.745 H_c$. More details about the critical fluctuations were presented by Kramer,[4,5] especially in relation to the problem of vortex nucleation. The question of metastability has also attracted the interest of the mathematical community,[6] and a detailed study of the instability due to one-dimensional fluctuations in type II superconductors was presented in Ref. 7. A similar analysis was performed for type I superconductors in Ref. 8, in which the results of earlier numerical[9,10] and analytical[11] investigations are confirmed and extended.

It is interesting to note that all the previous calculations of the superheating field were performed within the GL theory (with the exception of Ref. 3 in which the zero-temperature limit is also considered). This approach, however, is justified only near the critical temperature (i.e., if $T_c - T \ll T_c$), and a quantitative evaluation of the superheating field at low temperatures requires the use of the microscopic theory of superconductivity. Understanding the temperature dependence of the superheating field is important for practical applications; the maximum accelerating field of superconducting cavities used in particle accelerators is limited by the superheating field[12] and the optimal operational temperature lies well below T_c.[13]

In this work we consider, within the semiclassical approach of Eilenberger,[14] a clean type II superconductor occupying the half space $x > 0$ in the presence of an external magnetic field $\boldsymbol{H}_a$ parallel to the surface. We derive an expression for the thermodynamic potential valid at any temperature for $\kappa \gg 1$, which enables us to calculate the temperature dependence of the superheating field. As a result we find that in the limit $\kappa \to +\infty$ the ratio H_{sh}/H_c between superheating and thermodynamic critical fields is a nonmonotonic function of temperature which has a maximum at $T \approx 0.06 T_c$.

This paper is organized as follows: in Sec. II we briefly review the semiclassical theory of superconductivity and introduce our notation, while the derivation of the thermodynamic potential is presented in Sec. III. The conditions for the (meta)stability of the Meissner state are discussed in Sec. IV, and the superheating field is calculated in Sec. V. In Sec. VI we discuss the implications of our results for accelerator applications, and a brief summary is given in Sec. VII.

II. SEMICLASSICAL THEORY

In the semiclassical approach to superconductivity the superconducting system is described by the set of equations, the so-called Eilenberger equations, which are valid at any temperature[15] under the assumption that the Fermi wavelength λ_F is the smallest length scale characterizing the system. In practice this means that the semiclassical approximation usually applies to low-T_c superconductors, in which the zero-temperature coherence length $\xi_0 \gg \lambda_F$. This condition also implies the applicability at magnetic fields as high as H_{c2}. This technique is widely used to study the properties of hybrid superconducting devices; see, e.g., Ref. 16. From now on, we will use units such that Boltzmann constant $k_B=1$ and the Planck constant $\hbar=1$.

The Eilenberger equations are equations for the anomalous Green's functions $f(\omega_n,\boldsymbol{n},\boldsymbol{r})$ and $\bar{f}(\omega_n,\boldsymbol{n},\boldsymbol{r})$, which depend on the Matsubara frequencies $\omega_n=2\pi T(n+1/2)$, the position $\boldsymbol{r}$, and the unit vector $\boldsymbol{n}$ on the Fermi surface,

$$\{\omega_n + \boldsymbol{n}\cdot[\boldsymbol{\nabla} - i\boldsymbol{A}(\boldsymbol{r})]\} f(\omega_n,\boldsymbol{n},\boldsymbol{r}) = \Delta(\boldsymbol{r}) g(\omega_n,\boldsymbol{n},\boldsymbol{r}),$$

$$\{\omega_n - \boldsymbol{n}\cdot[\boldsymbol{\nabla} + i\boldsymbol{A}(\boldsymbol{r})]\} \bar{f}(\omega_n,\boldsymbol{n},\boldsymbol{r}) = \Delta^\dagger(\boldsymbol{r}) g(\omega_n,\boldsymbol{n},\boldsymbol{r}), \quad (1)$$

where the dagger denotes complex conjugation. The (normal) Green's function $g(\omega_n,\boldsymbol{n},\boldsymbol{r})$ is related to f and $\bar{f}$ via the constraint (suppressing all the arguments for brevity)

$$g^2 + f\bar{f} = 1. \quad (2)$$

These equations are to be solved together with the self-consistent equation for the complex order parameter $\Delta(\boldsymbol{r})$ and the Maxwell equation relating the magnetic field to the (super)current,

$$\Delta(\boldsymbol{r})\log\frac{T}{T_c} + 2\pi T\sum_n \left[\frac{\Delta(\boldsymbol{r})}{\omega_n} - \int \frac{d\boldsymbol{n}}{4\pi} f(\omega_n,\boldsymbol{n},\boldsymbol{r})\right] = 0, \quad (3)$$

$$\boldsymbol{\nabla}\times\boldsymbol{H} + i\frac{1}{\kappa_0^2} 2\pi T\sum_n \int \frac{d\boldsymbol{n}}{4\pi} 3\boldsymbol{n} g(\omega_n,\boldsymbol{n},\boldsymbol{r}) = 0, \quad (4)$$

with $\boldsymbol{H}=\boldsymbol{\nabla}\times\boldsymbol{A}$.

In Eqs. (1)–(4) we used as the unit of length the zero-temperature BCS coherence length

$$\xi_0 = \frac{v_F}{2\Delta_0}, \quad (5)$$

where v_F is the Fermi velocity and Δ_0 is the zero-temperature zero-field order parameter, which gives the energy unit. The vector potential $\boldsymbol{A}$ is rescaled by $\phi_0/2\pi\xi_0$ and the magnetic field $\boldsymbol{H}$ by $\phi_0/2\pi\xi_0^2$, with $\phi_0=\pi c/e$ as the flux quantum. These choices of units render all quantities dimensionless; for example, the BCS critical temperature is $T_c = e^{\gamma_E}/\pi \simeq 0.567$, where γ_E is Euler's constant. Finally, the dimensionless parameter κ_0, the only independent parameter remaining after the units are chosen, is defined in analogy with the GL parameter κ as

$$\kappa_0 = \frac{\lambda_0}{\xi_0}, \quad (6)$$

where the zero-temperature penetration depth is

$$\frac{1}{\lambda_0^2} = \frac{8\pi}{3}\left(\frac{2\pi\xi_0}{\phi_0}\right)^2 \nu\Delta_0^2, \quad (7)$$

with ν as the density of states at the Fermi energy.

In writing Eq. (1) higher-order terms in the magnetic field are neglected which give rise to diamagnetic effects.[17] This is a good approximation if

$$\omega_c \ll T, \quad (8)$$

where $\omega_c=eH/mc$ is the cyclotron frequency. This condition can be rewritten as

$$\frac{\lambda_F}{\xi_0}\frac{1}{\kappa_0}\frac{H}{H_c(0)} \ll \frac{T}{T_c}. \quad (9)$$

In low T_c, strong type II superconductors, the first two factors on the left-hand side are both small parameters. In what follows we consider magnetic fields H smaller than the zero-temperature critical field $H_c(0)$; therefore, this approximation is justified down to very low temperatures.[18]

III. THERMODYNAMIC POTENTIAL

Equations (1), (3), and (4) are the Euler-Lagrange equations obtained by varying the following functional[14] with respect to $\bar{f}$ and f, $\Delta^\dagger$, and $\boldsymbol{A}$, respectively:

$$\Omega = \nu\int d^3r\Bigg\{\frac{\kappa_0^2}{3}[\boldsymbol{H}(\boldsymbol{r})-\boldsymbol{H}_a]^2 + |\Delta(\boldsymbol{r})|^2\log\left(\frac{T}{T_c}\right) + \int(dn)\Bigg[\frac{|\Delta(\boldsymbol{r})|^2}{\omega_n} - \Delta^\dagger(\boldsymbol{r})f - \bar{f}\Delta(\boldsymbol{r}) - 2\omega_n(g-1) - g\boldsymbol{n}\cdot\left(\boldsymbol{\nabla}\log\frac{f}{\bar{f}} - 2i\boldsymbol{A}(\boldsymbol{r})\right)\Bigg]\Bigg\}, \quad (10)$$

with g as implicitly defined by Eq. (2), $\boldsymbol{H}_a$ as the applied field which we assume uniform, and

$$\int(dn) \equiv 2\pi T\sum_n \int \frac{d\boldsymbol{n}}{4\pi}. \quad (11)$$

The functional Ω in Eq. (10) is not the thermodynamic potential; however, for any given $\Delta(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$ Eq. (10) gives the difference between the potentials in the superconducting and normal states once the solutions to Eq. (1) for f and $\bar{f}$ are substituted into it.

As a first step in solving Eq. (1), we recall that the order parameter can be assumed as real; more precisely, the (gradient of the) phase of the order parameter can be collected together with the vector potential into a gauge-invariant quantity. All other quantities become gauge invariant as well, and the new vector potential is proportional to the supercurrent velocity. This means that in the Meissner state $\boldsymbol{A}$ must vanish deep into the superconductor; the same holds for its component perpendicular to the surface, as no current leaves the superconductor. Moreover, $\boldsymbol{H}=\boldsymbol{H}_a$ at the surface. These arguments determine the boundary conditions for $\boldsymbol{A}$; further boundary conditions are discussed at the end of this section.

With a real order parameter, it is convenient to introduce the sum and difference of the anomalous Green's functions,

$$s = f + \bar{f}, \quad d = f - \bar{f}. \tag{12}$$

Then, Eq. (2) can be rewritten as

$$g^2 = 1 - \frac{1}{4}(s^2 - d^2). \tag{13}$$

Taking the difference between the equations in Eq. (1) and solving for d in terms of s, we obtain

$$d = -\frac{\boldsymbol{n} \cdot \boldsymbol{\nabla} s}{\Omega_n}, \tag{14}$$

where we introduced the short hand notation

$$\Omega_n = \omega_n - i\boldsymbol{n} \cdot \boldsymbol{A}. \tag{15}$$

Our discussion so far is valid for any κ_0, but to find an explicit expression for the thermodynamic potential as a functional of $\Delta(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$, we look for an approximate solution to Eq. (1) for $\kappa_0 \gg 1$ and at arbitrary temperature. To find a suitable approximate expression for s, we note that a rescaling $\boldsymbol{r} \to \kappa_0 \boldsymbol{r}$ only affects the gradient terms in Eq. (1), so that an expansion in the small parameter $1/\kappa_0$ is equivalent to a gradient expansion. To zero order we neglect the gradient terms, drop d in Eq. (13), and using the sum of Eq. (1) arrive at

$$s^{(0)} = \frac{2\Delta}{\sqrt{\Omega_n^2 + \Delta^2}},$$

$$g^{(0)} = \frac{\Omega_n}{\sqrt{\Omega_n^2 + \Delta^2}}, \tag{16}$$

which for $\boldsymbol{A}=0$ correctly reduce to the standard result[16] for a bulk superconductor in the absence of magnetic field. Hereinafter, due to the above-mentioned rescaling, λ_0 is the unit of length.

To calculate the next order in the expansion, we define [see Eq. (14)]

$$d^{(1)} = -\frac{1}{\kappa_0}\frac{\boldsymbol{n} \cdot \boldsymbol{\nabla} s^{(0)}}{\Omega_n}. \tag{17}$$

From Eq. (13) we obtain

$$g \simeq g^{(0)} + g^{(2)},$$

$$g^{(2)} = \frac{(d^{(1)})^2}{8g^{(0)}} - \frac{s^{(0)}}{4g^{(0)}} s^{(2)}, \tag{18}$$

where $s^{(2)}$ is the next nontrivial order in the expansion for s (i.e., $s \simeq s^{(0)} + s^{(2)}$), which is again found from the sum of Eq. (1),

$$\kappa_0^2 s^{(2)} = \frac{\Delta}{4}\frac{(\boldsymbol{n} \cdot \boldsymbol{\nabla} s^{(0)})^2}{\Omega_n^2}\frac{1}{S_n} + \frac{(\boldsymbol{n} \cdot \boldsymbol{\nabla})^2 s^{(0)}}{S_n^2} + \frac{(\boldsymbol{n} \cdot \boldsymbol{\nabla})\hat{A}}{\Omega_n}\frac{\boldsymbol{n} \cdot \boldsymbol{\nabla} s^{(0)}}{S_n^2} \tag{19}$$

with

$$S_n = \sqrt{\Omega_n^2 + \Delta^2}. \tag{20}$$

It turns out that this expression, however, is not needed to obtain the thermodynamic potential, as its contributions to it cancel out. This can be checked by substituting Eqs. (12) and (13) into Eq. (10) and using the approximate expressions in Eqs. (16)–(18). After an integration by parts (and dropping the resulting surface term), to lowest nontrivial order in the gradient expansion, the thermodynamic potential as a functional of Δ and $\boldsymbol{A}$ is

$$\Omega = \nu \int d^3r \Bigg\{ \frac{1}{3}(\boldsymbol{\nabla} \times \boldsymbol{A} - \boldsymbol{H}_a)^2 + \Delta^2 \log\left(\frac{T}{T_c}\right) + \int (dn) \Bigg[\frac{\Delta^2}{\omega_n} - 2(\sqrt{\Omega_n^2 + \Delta^2} - \omega_n) + \frac{1}{\kappa_0^2}\frac{\sqrt{\Omega_n^2 + \Delta^2}}{4\Omega_n^2}(\boldsymbol{n} \cdot \boldsymbol{\nabla} s^{(0)})^2 \Bigg] \Bigg\}. \tag{21}$$

This expression is one of our main results and is the starting point to study the metastability of the Meissner state; see Sec. IV. It can be considered as an extension of the GL approach, and as a check, we show in Sec. III A that Eq. (21) reduces to the known GL potential in the appropriate limit. It is interesting to compare the present result with other extensions of the GL theory in the literature,[19–23] where the expansion is performed with respect to the covariant derivative [i.e., the gauge-invariant operator $\nabla - 2ie\boldsymbol{A}$][24]. In the present notation, this amounts to supplementing the already performed gradient expansion with an expansion over $\boldsymbol{A}$; at lowest nontrivial order the published results are recovered. As the field increases, however, $\boldsymbol{A}$ increases as well and this additional expansion is not reliable. In the local limit considered here the order-parameter amplitude spatial profile is determined primarily by the depairing effect of the supercurrent, which is taken into account exactly, while the additional gradient terms are suppressed by the small parameter $1/\kappa_0^2$.[25]

A. Ginzburg-Landau limit

As remarked in Sec. I, the GL approach is valid near the critical temperature. More generally, near a second-order phase transition the order parameter is small and an expansion of the thermodynamic potential in powers of the small parameter $\Delta/2\pi T$ becomes viable. To perform this expansion in the present case, we introduce the rescaled vector potential

$$\tilde{\boldsymbol{A}} = \sqrt{\frac{2}{3}}\frac{\boldsymbol{A}}{\Delta(T)}. \tag{22}$$

Here $\Delta(T)$ is the value of the order parameter at temperature T in zero magnetic field, which by definition satisfies the equation

$$\log\left(\frac{T}{T_c}\right) + 2\pi T \sum_n \left[\frac{1}{\omega_n} - \frac{1}{\sqrt{\omega_n^2 + \Delta(T)^2}}\right] = 0 \tag{23}$$

obtained by minimizing Eq. (21) with $\boldsymbol{A} = \boldsymbol{H}_a = 0$. The expansion of the above equation near T_c leads to the well-known[26] approximate expression

$$\frac{\Delta(T)}{2\pi T} \approx \sqrt{\frac{2}{\zeta}}\sqrt{1-\frac{T}{T_c}}, \tag{24}$$

where

$$\zeta = \sum_n \frac{1}{(n+1/2)^3} = -\psi''\left(\frac{1}{2}\right) = 7\zeta(3). \tag{25}$$

The rescaling in Eq. (22) is equivalent to normalizing the field with respect to the temperature-dependent thermodynamic critical field rather than its zero-temperature value; the scaling is to be applied to the external field $\boldsymbol{H}_a$ as well. Similarly, we define the normalized order parameter

$$\psi(\boldsymbol{r}) = \frac{\Delta(\boldsymbol{r})}{\Delta(T)} \tag{26}$$

and introduce the temperature-dependent penetration depth

$$\lambda(T) = \frac{\lambda_0}{\sqrt{\zeta}}\frac{2\pi T}{\Delta(T)} \approx \frac{\lambda_0}{\sqrt{2}}\frac{1}{\sqrt{1-T/T_c}} \tag{27}$$

as the unit of length. It should be stressed that all the temperature dependencies introduced in this subsection are valid only in the vicinity of the critical temperature; see also Ref. 26.

Substituting the definitions in Eqs. (22) and (26) into Eq. (21), expressing lengths via $\lambda(T)$, and expanding in powers of $\Delta(T)/2\pi T \ll 1$, the lowest-order term is

$$\Omega_{\rm GL} = \nu\zeta\frac{\Delta^4(T)}{(2\pi T)^2}\int d^3r\left\{\frac{1}{2}(\boldsymbol{\nabla}\times\tilde{\boldsymbol{A}}-\boldsymbol{H}_a)^2 - \frac{1}{2}\psi^2 + \frac{1}{2}\tilde{A}^2\psi^2 + \frac{1}{4}\psi^4 + \frac{1}{2\kappa_{\rm GL}^2}(\boldsymbol{\nabla}\psi)^2\right\}. \tag{28}$$

The GL parameter $\kappa_{\rm GL}$ is proportional to κ_0,

$$\kappa_{\rm GL} = \sqrt{\frac{3}{2\zeta}}\kappa_0 \approx 0.42\kappa_0. \tag{29}$$

It can be written as the ratio

$$\kappa_{\rm GL} = \frac{\lambda(T)}{\xi(T)} \tag{30}$$

between the temperature-dependent penetration depth [Eq. (27)] and coherence length

$$\xi(T) = \sqrt{\frac{2}{3}}\frac{2\pi T}{\Delta(T)}\xi_0. \tag{31}$$

Inside the curly brackets in Eq. (28) one can recognize the GL free energy in its dimensionless form; see, e.g., Refs. 4 and 8. The last term, in particular, represents the energy associated with the spatial variation of the amplitude of the order parameter. Due to this term the appropriate boundary condition at the surface is $\psi'=0$, where the prime indicates the derivative along the normal to the surface. The similar term in Eq. (21) has a more complicated dependence on both the order parameter and the vector potential; for this reason the issue of the boundary condition for Δ demands further investigation beyond the scope of the present work; for example, the addition of surface terms [such as those discarded in obtaining Eq. (21)] may be necessary; see, e.g., Ref. 27. In what follows, however, we will concentrate on the limit $\kappa_0 \to\infty$; in this case the last term in Eq. (21) is neglected, the order parameter is determined by a "local" equation (rather than a differential one—see Sec. IV), and no boundary conditions are required for $\Delta(\boldsymbol{r})$.

IV. STABILITY CONDITION

In Sec. III we have derived an approximate expression [Eq. (21)] for the thermodynamic potential valid at large κ_0. A standard procedure can be now applied to study the properties of this functional; for example, the self-consistent equation for Δ and the Maxwell equation are found by taking the variations with respect to Δ and $\boldsymbol{A}$, respectively. Considering from now on only the lowest-order contributions in $1/\kappa_0$ [i.e., neglecting the last term on the right-hand side of Eq. (21)] and using Eq. (23), we find

$$\int (dn)\left[\Delta\left(\frac{1}{\sqrt{\omega_n^2+\Delta(T)^2}} - \frac{1}{\sqrt{\Omega_n^2+\Delta^2}}\right)\right] = 0, \tag{32}$$

which gives, via definition (15), a local relation between order parameter and vector potential, and

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{A} - i\int (dn)3\boldsymbol{n}\frac{\Omega_n}{\sqrt{\Omega_n^2+\Delta^2}} = 0. \tag{33}$$

Solutions to Eqs. (32) and (33) are (meta)stable only if they are a minimum of Ω; i.e., if its second variation is positive. To investigate the stability, let us parametrize Δ and $\boldsymbol{A}$ as

$$\Delta = \Delta_s + \eta, \quad \boldsymbol{A} = \boldsymbol{A}_s + \boldsymbol{a}, \tag{34}$$

where Δ_s and $\boldsymbol{A}_s$ satisfy Eqs. (32) and (33). Expanding Eq. (21) for small η and $\boldsymbol{a}$, the second variation is

$$\delta^2\Omega = \nu\int d^3r\left\{\int (dn)\left[\frac{\Delta_s^2}{(\Omega_s^2+\Delta_s^2)^{3/2}}(\eta^2 + (\boldsymbol{n}\cdot\boldsymbol{a})^2) - 2i\frac{\Omega_s\Delta_s}{(\Omega_s^2+\Delta_s^2)^{3/2}}\eta(\boldsymbol{n}\cdot\boldsymbol{a})\right] + \frac{1}{3}(\boldsymbol{\nabla}\times\boldsymbol{a})^2\right\} \tag{35}$$

with $\Omega_s=\omega_n - i\boldsymbol{n}\cdot\boldsymbol{A}_s$. In the absence of magnetic field (i.e., $\boldsymbol{A}_s=0$) the superconducting state is stable for any $T<T_c$; after integrating over $\boldsymbol{n}$ the last term in square brackets in Eq. (35) vanishes, so that $\delta^2\Omega>0$ for any fluctuation as long as $\Delta_s>0$. As the field increases, however, the sign of $\delta^2\Omega$ changes, by definition, when the superheating field is reached. Therefore, to find $H_{\rm sh}$ we look for nontrivial fluctuations $\eta, \boldsymbol{a}\neq 0$ such that $\delta^2\Omega[\Delta_s,\boldsymbol{A}_s]=0$.

In the geometry under consideration (i.e., superconductor in the $x>0$ half space and $\boldsymbol{H}_a \parallel z$), the solution to Eqs. (32) and (33) is parametrized as

$$\Delta_s = \Delta_s(x), \quad \boldsymbol{A}_s = [0, A_y(x), 0]. \tag{36}$$

Then, as shown by Kramer[4] in the GL limit, the fluctuations can be taken in the following form:

$$\eta = \tilde{\eta}(x,k)\cos(ky),$$

$$\boldsymbol{a} = [\tilde{a}_x(x,k)\sin(ky), \tilde{a}_y(x,k)\cos(ky), 0]. \quad (37)$$

Substituting Eq. (37) into Eq. (35) and minimizing with respect to $\tilde{\eta}$ and $\tilde{a}_x$, we find

$$\tilde{\eta} = \frac{G}{F_0}\tilde{a}_y, \quad (38)$$

$$\tilde{a}_x = \frac{k}{3F_x + k^2}\tilde{a}'_y, \quad (39)$$

with prime denoting the differentiation with respect to x,

$$G = \int (dn)\frac{i\Omega_s\Delta_s n_y}{(\Omega_s^2 + \Delta_s^2)^{3/2}} \quad (40)$$

and

$$F_i = \int (dn)\frac{\Delta_s^2 n_i^2}{(\Omega_s^2 + \Delta_s^2)^{3/2}}. \quad (41)$$

Here $n_0 = 1$ and we note the property $|F_i| \leq 1$. With these definitions, we obtain for the second variation (up to a numerical prefactor)

$$\delta^2\Omega \propto \int_0^\infty dx\left[\frac{F_x}{3F_x + k^2}(\tilde{a}'_y)^2 + \frac{F_0F_y - G^2}{F_0}(\tilde{a}_y)^2\right]. \quad (42)$$

The first term on the right-hand side of Eq. (42) gives always a positive contribution to the second variation, but as we now argue it can be neglected in the large κ_0 limit. Clearly, the larger k is the smaller this contribution becomes; on the other hand, the second variation of the last term in Eq. (21), which we have neglected, would schematically contribute a (positive) term proportional to k^2/κ_0^2. Therefore, the optimal value is $k \sim \sqrt{\kappa_0}$, which is in agreement with the GL result of Ref. 4, and the two terms both give contributions of $\sim 1/\kappa_0$, which we neglect as $\kappa_0 \to +\infty$. As a consequence, the superheating field is determined by the vanishing of the coefficient of the second term on the right-hand side of Eq. (42), i.e.,

$$F_0F_y - G^2 = 0. \quad (43)$$

In Sec. V we use this condition together with Eqs. (32) and (33) to calculate the superheating field.

V. SUPERHEATING FIELD

The stability analysis of Sec. IV gives a conceptually simple procedure to find the superheating field; we should first solve Eqs. (32) and (33) to find the profile [cf. Eq. (36)] of the (meta)stable superconducting state for a given applied magnetic field, and then find H_a such that the condition for the instability threshold in Eq. (43) is satisfied. The task of solving the nonlinear differential equation (33), however, makes this route difficult in practice. An alternative approach is based on the observation that the combination

$$H_s^2 - 3\int (dn)\left[\frac{\Delta_s^2}{\sqrt{\Omega_s^2 + \Delta_s^2}} - 2(\sqrt{\Omega_s^2 + \Delta_s^2} - \omega_n)\right], \quad (44)$$

where $H_s = \boldsymbol{\nabla} \times \boldsymbol{A}_s$, is constant throughout the superconductor,[28] as can be checked using Eqs. (32) and (33). Taking into account the boundary conditions, the form of the solution in Eq. (36), and the expression

$$H_c^2(T) = 6\pi T\sum_n\left[2(\sqrt{\omega_n^2 + \Delta(T)^2} - \omega_n) - \frac{\Delta(T)^2}{\sqrt{\omega_n^2 + \Delta(T)^2}}\right] \quad (45)$$

for the critical field, we find (cf. Ref. 3)

$$H_a^2 = H_c^2 + 6\pi T\sum_n\left[2\omega_n + \frac{1}{A_0}\mathrm{Im}(\Omega_0\sqrt{\Omega_0^2 + \Delta_{s0}^2})\right], \quad (46)$$

where $A_0 = A_y(0)$, $\Omega_0 = \omega_n - iA_0$, and $\Delta_{s0} = \Delta_s(0)$. The above equation relates the applied field to the values of the order parameter and vector potential at the surface. At the superheating field, these two quantities can be found by solving the local Eqs. (32) and (43). This can be done analytically in the limiting cases $T \to T_c$ and $T \to 0$, as we now show.

A. Limiting cases

In the GL ($T \to T_c$) limit Eqs. (32) and (43) reduce to, respectively,

$$\Delta_{s0}^2 + \frac{2}{3}A_0^2 = \Delta(T)^2, \quad (47)$$

$$\frac{1}{3}\Delta_{s0}^4 - \frac{4}{9}A_0^2\Delta_{s0}^2 = 0. \quad (48)$$

Solving these equations we find $\Delta_{s0} = \sqrt{2/3}\Delta(T)$. Defining

$$\tilde{H} \equiv \frac{H_{\mathrm{sh}}}{H_c} \quad (49)$$

and substituting the result into the GL limit of Eq. (46)

$$\tilde{H}^2 = 1 - \frac{\Delta_{s0}^4}{\Delta(T)^4}, \quad (50)$$

we arrive at $\tilde{H} = \sqrt{5}/3 \simeq 0.745$, which is in agreement with Refs. 3 and 4. This is not surprising since we showed in Sec. III A the reduction of our thermodynamic potential [Eq. (21)] to the GL one [Eq. (28)] in this limit.

In the opposite limit $T \to 0$ and using the notation $\lambda = \Delta_{s0}/A_0$, Eqs. (32) and (43) become[29]

$$\log(\Delta_{s0}) = 0 \quad (A_0 < \Delta_{s0}),$$

$$\log[A_0(1 + \sqrt{1-\lambda^2})] - \sqrt{1-\lambda^2} = 0 \quad (\Delta_{s0} < A_0 < e/2), \quad (51)$$

$$[1 - \sqrt{1-\lambda^2}]\frac{1}{3}[1 - \sqrt{1-\lambda^2}(1+2\lambda^2)] - [\lambda\sqrt{1-\lambda^2}]^2 = 0, \quad (52)$$

while Eq. (46) can be written as

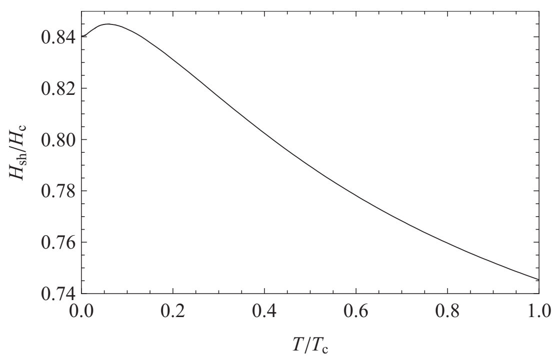

FIG. 1. Temperature dependence of the ratio H_{sh}/H_c. Note the nonmonotonic behavior with a maximum at low temperature $T \approx 0.06T_c$.

$$\tilde{H}^2 = 1 - \frac{2}{3}A_0^2\left[\frac{1}{2} - \frac{3}{2}(1-\lambda^2) + (1-\lambda^2)^{3/2}\right]. \qquad (53)$$

Substitution of the solution to Eqs. (51) and (52) into the latter expression finally gives $\tilde{H} \simeq 0.840$, as found in Ref. 3.

B. Temperature dependence

Having verified that the known limiting results are reproduced with our approach, we now consider the temperature dependence of $\tilde{H}$. Near the two limiting temperatures we can in principle calculate temperature-dependent corrections by further expanding over the small parameters $\Delta/2\pi T$ close to T_c and T/Δ_0 as $T \to 0$. Instead, to obtain the behavior at arbitrary temperature, we resort to a numerical approach. Following the same strategy as in Sec. V A we first solve numerically the system composed of Eqs. (32) and (43) to find A_0 and Δ_{s0}; then, we substitute the result into Eq. (46). In this way, we obtain the curve presented in Fig. 1. Interestingly, we find a nonmonotonic dependence of $\tilde{H}$ on temperature with a maximum of $\tilde{H}^{\max} \approx 0.845$ at $T \approx 0.06T_c$. A nonmonotonic behavior is also found in $H_{sh}(T)$ shown in Fig. 2. Taking into account the decrease in $H_c(T)$ with increasing temperature, $H_{sh}(T)$ acquires its maximum value of $H_{sh}^{\max}$ $\approx 0.843H_c(0)$ at the lower temperature $T \approx 0.04T_c$, see the inset of Fig. 2. Since $H_c^2(0) = 4\pi\nu\Delta_0^2$ depends only on material properties, this implies that in the London limit there is an optimal temperature at which the superheating field is the highest possible for a given material.

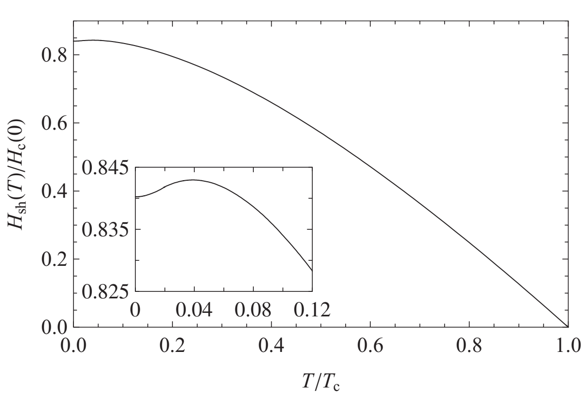

FIG. 2. Temperature dependence of H_{sh} normalized by the zero-temperature critical field. The nonmonotonic behavior with a maximum at $T \approx 0.04T_c$ is evident in the inset, which zooms in on the low-temperature region.

VI. IMPLICATIONS FOR ACCELERATOR DESIGN

The rf cavities used in modern particle accelerators, both for high-energy physics and as x-ray sources, are made of superconducting niobium. The best Nb cavities are operated in the metastable region well above H_{c1}. The superheating field H_{sh} provides an upper bound for the maximum particle acceleration that a given cavity can produce,[12] and the operating point for the best cavities is approaching the theoretical limit provided by the GL theory, when the latter is extrapolated to the operating condition ($T \sim 0.2T_c$) where it is not valid. Niobium is not a high-κ material, and our theory cannot be directly applied to it, but Fig. 1 would indicate that at $T/T_c \sim 0.2$ the true superheating field would be 11% higher than the GL estimate for a high-κ material. A change in the theoretical upper bound of this magnitude would have significant implications for future attempts to improve the material processing of existing Nb-based cavities. In principle, a numerical solution of the linear stability problem for the Eilenberger equations should be possible (albeit challenging) for all values of κ, including $\kappa \sim 1$—a calculation of direct significance to current technological applications.

There are several other superconducting materials which appear potentially promising as eventual replacements for Nb in future accelerator applications, all of which have significantly higher κ and hence are potentially better described by our London limit calculation. For example, if run at the current operating temperature of 2 K, cavities made of Nb_3Sn or MgB_2 would be near the peak of H_{sh}/H_c in Fig. 1, and hence, our calculation would suggest a peak field 13% higher than that provided by the GL theory. Using a current design for the superconducting cavity, our result for H_{sh} suggests a theoretical upper bound for the accelerating field of 200 MV/m, a factor of 4 larger than the operating fields of the best Nb-based cavities. Material difficulties have so far kept high-temperature copper-oxygen-based superconductors from being useful in these applications, but new high-T_c materials, e.g., iron pnictides,[30] may provide more forgiving material properties. A quantitative estimate of the superheating field in these materials, however, may demand calculations that incorporate effects that go beyond the semiclassical analysis of the present work. For example, a complete description of superconductivity in MgB_2 requires an Eliashberg-type calculation,[31] with material properties extracted from a density-functional electronic structure calculation.[32]

We point out that our result is in sharp contrast with the commonly used heuristic $H_{sh} \sim H_c/\kappa$ of Yogi *et al.*[33] This heuristic, termed as the "line nucleation model," is not a linear stability calculation but an energy balance argument that gives a nonsensical estimate $H_{sh} < H_{c1}$ for large κ. The formula's success in describing experiments[33] suggests that there may be nucleation mechanisms (perhaps disorder mediated) that become more difficult to control in high-κ materials but it should be viewed as an experimental extrapolation, rather than a theoretical bound, in guiding the exploration of new materials.

VII. SUMMARY AND OPEN PROBLEMS

In this paper we have revisited the problem of evaluating the superheating field for type II superconductors, in particular with regard to its dependence on temperature. To extend previous calculations[2,4–6] based on the Ginzburg-Landau theory, which is restricted to temperatures close to the critical one, we have employed the semiclassical approach in order to derive an approximate expression for the thermodynamic potential [Eq. (21)] valid at large values of the Ginzburg-Landau parameter $\kappa \gg 1$. From this expression we have calculated, in the limit $\kappa \to +\infty$, the temperature dependence of the ratio between the superheating and critical fields which is presented in Fig. 1. The relevance of our results to applications in particle accelerators is discussed in Sec. VI.

Many natural extensions of this work come to mind, beyond and in connection with those already mentioned in Sec. VI. For example, it would be interesting to study in more detail the critical variations, as done in Ref. 4 in the GL limit, and the finite κ corrections. In our calculations we have considered the simplest possible case, namely, a clean superconductor with a spherical Fermi surface. However, the semiclassical theory can easily accommodate anisotropies in the Fermi surface. The effect of bulk impurities can also be incorporated in the formalism. In superconducting cavities in the presence of rf fields, various mechanisms for the breakdown of superconductivity are associated with characteristics of the surface, e.g., the presence of surface impurities or steps caused by grain boundaries.[12] Moreover surface properties, namely, specular vs diffuse reflection, are known to affect the electromagnetic response of impure superconductors.[34] Therefore, it would be important to explore theoretically the impact of surface imperfections on the superheating field.[35]

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[25] This approximation breaks down if the order parameter changes by an amount of order Δ_0 (i.e., order 1 in our notation) over a length scale comparable to ξ_0 ($=1/\kappa_0$ in our units), so that $|\nabla\Delta| \sim k_0$. This is the case, for example, near a vortex core at low temperatures (but not at sufficiently high temperatures, since the change in Δ is approximately limited by $\Delta(T) \le \Delta_0$ over a length scale of order $\xi(T) \ge \xi_0$) or for nonuniform applied fields with large variations over the same length scale (i.e., such that $\xi_0 |\nabla H| / H \sim 1$).
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[28] This can be justified by treating x as a time coordinate, $\boldsymbol{A}$ and Δ as generalized coordinates, and Ω as an action; then Eq. (44) corresponds to the energy conservation law.
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